Comment on Strings in $AdS_3 \times S^3 \times S^3 \times S^1$ at One Loop

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This paper studies semiclassical strings in $AdS_3 \times S^3 \times S^3 \times S^1$ using the algebraic curve. Calculating one-loop corrections to the energy of the giant magnon fixes the constant term c in the expansion of the coupling $h(\lambda)$. Comparing these to similar corrections for long spinning strings gives a prediction for the one-loop term f_1 in the expansion of the cusp anomalous dimension $f(h)$, for all α (where $\alpha \to 1$ is the $AdS_3 \times S^3 \times T^4$ limit). For these semiclassical mode sums there is a similar choice of regularisation prescriptions to that encountered in $AdS_4 \times CP^3$. However at $\alpha \neq \frac{1}{2}$ they lead to different values of f_1 and are therefore not related by a simple change of the coupling. The algebraic curve is also used to calculate various finite-size corrections for giant magnons, which are well-behaved as $\alpha \to 1$, and can be compared to the recently published S-matrices.

1. Introduction

The usual starting point for discussing integrable strings in $AdS_5 \times S^5$ is the Metsaev–Tseytlin coset action [1], where classical integrability follows from the fact that the coset is a Riemannian symmetric space [2]. This is the strong-coupling end of the best-studied example of AdS/CFT, and the integrable structure now extends to all values of the 't Hooft coupling λ [3]. The same statements are true for the second-best-studied example, with strings on $AdS_4 \times CP^3$ [4].

One of the challenges of studying integrability in backgrounds such as the $AdS_3 \times S^3 \times T^4$ arising from the D1-D5 system [5] is the presence of flat directions, and hence massless modes, which are not captured by the coset action. This is also true of the $AdS_3 \times S^3 \times S^3 \times S^1$ background studied here, which has a parameter $\alpha = \cos^2 \phi$ controlling the relative size of the two 3-spheres [6], and hence the masses of the modes in these directions. One of the reasons this space is interesting is that in the limit $\alpha \to 1$ one S^3 decompactifies to give (when combined with the S^1) a T^4 factor. In this limit two more bosonic modes become massless, and it is hoped that we may learn about how to handle massless modes by studying this process.

There is no known CFT_2 gauge theory dual for general α [7], although at $\alpha = 1$ there is a symmetric product-space CFT [8] as well as more recently a spin chain [9] and some work on magnons [10,11]. At $\alpha \neq 1$ there is much recent work on integrability [12–19] perhaps the highlight of which is a conjectured all-order Bethe ansatz for all α [15]. This similarly omits the massless modes (as well as the heavy modes, discussed below) but has a good $\alpha \to 1$

limit. As we have learned from the $AdS_4 \times CP^3$ correspondence, the Bethe equations give the spectrum in terms of a coupling $h(\lambda)$ whose relationship to λ (or more precisely here to R^2/α') must be found experimentally [20–22]. In this case the strong-coupling expansion is

$$h = 2g + c + \mathcal{O}\Big(\frac{1}{g}\Big) \qquad\qquad g = \frac{R^2}{4\pi\alpha'} = \frac{\sqrt{\lambda}}{4\pi} \gg 1.$$

Here R is the radius of the AdS_3 part of the spacetime, and the spheres' $R_\pm$ are as follows:

$$ds^2 = R^2\, ds^2_{AdS_3} + \frac{R^2}{\cos^2\phi}\, ds^2_{S^3_+} + \frac{R^2}{\sin^2\phi}\, ds^2_{S^3_-} + R^2\, d\psi^2. \tag{1}$$

The BMN point particle (which is the spin chain vacuum) has momentum on both of the spheres: the solution is $\vartheta_+ + \vartheta_- = \tau$ in terms of the two azimuthal angles. The two bosonic massless modes are are fluctuations in ψ and in $\vartheta_\perp = \tan\phi\, \vartheta_+ + \cot\phi\, \vartheta_-$. Both of these are absent from the coset model $D(1,2;\alpha)^2/SU(1,1)\times SU(2)^2$. The algebraic curve for this was introduced by Babichenko, Stefański and Zarembo in [12].

The goal of this paper is to use this to calculate (or to guide the calculation of) semiclassical energy corrections for various classical string solutions. Such corrections have played an important role in the past [23]. As in $AdS_4 \times CP^3$ there is a distinction between light modes, which are excitations of the Bethe equations, and heavy modes which are in some senses composite objects, and because of this there are similar issues of regularisation [22, 24–29]. However (as we will see) in this case they *cannot* always be absorbed into a modification of the coupling constant.

The two classical systems to be studied are long spinning strings in AdS_3, and giant magnons in S^3. In both cases the classical solutions are identical to those in $AdS_5 \times S^5$, apart from momentum on some S^1 factors.

- Giant magnons have [30, 15]

$$\begin{aligned}\Delta - J' &= \sqrt{m_r^2 + 4h^2 \sin^2\frac{p}{2}} \\ &= 4g \sin\frac{p}{2} + 2c\sin\frac{p}{2} + \mathcal{O}\Big(\frac{1}{g}\Big)\end{aligned} \tag{2}$$

  where the mass m_r depends on which sphere the solution lives in:

$$m_1 = \sin^2\phi = 1-\alpha\,, \qquad m_3 = \cos^2\phi = \alpha\,.$$

  Using the algebraic curve formalism of [12, 14] to calculate the one-loop correction to the energy δE allows us to find c. As in $AdS_4 \times CP^3$ the result depends on the regularisation used; with a cutoff on the physical energy it is

$$c_{\text{phys}} = \frac{\alpha\log\alpha + (1-\alpha)\log(1-\alpha)}{2\pi}. \tag{26}$$

- Long spinning strings have [31]

$$\begin{aligned}\Delta - S = f(\lambda)\log S, \qquad f &= 2h + f_1 + \mathcal{O}\Big(\frac{1}{h}\Big) \\ = 4g\log S + \delta\Delta &+ \mathcal{O}\Big(\frac{1}{g}\Big).\end{aligned} \tag{3}$$

  Mode frequencies for these strings were calculated by [16], and will be used here to discuss the dependence of the one-loop term $\delta\Delta$ on the regularisation prescription. Unlike the giant magnons, the relevant term from integrability f_1 depends on the one-

loop part of the dressing phase $\sigma_{\rm HL}$ [32–35, 21].

Comparing results from these two systems gives a prediction for f_1 which indicates that the dressing phase must be different to that seen in $AdS_5 \times S^5$ and $AdS_4 \times CP^3$. This prediction appears to depend on the regularisation used, but demanding that it is well-behaved as $\alpha \to 1$ rules out the cutoff in the spectral plane.

The calculation of δE for the giant magnon can also easily be extended to include finite-size corrections. The Lüscher F-terms calculated this way can normally be compared term-by-term to certain (diagonal) elements of the S-matrix, which was until this week unknown.

However yesterday and two days before, two papers appeared each aiming to derive the S-matrix for $AdS_3 \times S^3 \times S^3 \times S^1$ [36, 37]. A preliminary comparison of our results shows agreement with the elements in both of these, modulo some issues of phases.

Outline

Section 2 reviews the setup of the algebraic curve, and section 3 the various cutoff prescriptions. Section 4 uses all of this for giant magnons. Section 5 looks at summing frequencies for the spinning string, and what we can learn from the comparison. Section 6 has a summary and 13 comments.

Appendix A looks at finite-size corrections (classical and one-loop) and the comparison to the proposed S-matrices. Appendix B looks briefly at the algebraic curve for $AdS_3 \times S^3 \times T^4$, and the matching of corrections in this limit. Appendix C deals with classical giant magnons in the sigma-model.

2. Algebraic Curves for $AdS_3 \times S^3 \times S^3$ sans S^1

The algebraic curve, or finite-gap method, is a way of writing classical string solutions as Riemann surfaces [38]. The Lax connection (which depends on the spectral parameter $x \in \mathbb{C}$) is integrated around the worldsheet, and the path-ordered exponential of this is the monodromy matrix, whose eigenvalues are $e^{\pm i p_\ell}$ with $p_\ell(x)$ called quasimomenta. These contain essentially all the information about the solution.[1] This formalism has been especially useful for semiclassical quantisation, where vibrational modes are represented by small perturbations of the quasimomenta [40–42].

The setup described in this section is largely from [12], see also [14]. It starts from the Cartan matrix for $d(2,1;\alpha)^2$. Since this is a continuous family of distinct Lie super-algebras, $A_{\ell m}$ has non-integer entries:

$$A = \begin{bmatrix} 4\sin^2\phi & -2\sin^2\phi & 0 \\ -2\sin^2\phi & 0 & -2\cos^2\phi \\ 0 & -2\cos^2\phi & 4\cos^2\phi \end{bmatrix} \otimes 1_{2\times 2}\,. \tag{4}$$

For each Cartan generator Λ_ℓ there is a quasimomentum $p_\ell(x)$, where $\ell = 1, 2, 3, \bar{1}, \bar{2}, \bar{3}$. In addition to A, we also need to know the matrix S which gives the inversion symmetry (and in general the effect of the $\mathbb{Z}_4$ symmetry). In this case it exchanges the left and right copies:

$$p_\ell(\tfrac{1}{x}) = S_{\ell m} p_m(x), \qquad S = 1_{3\times 3} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5}$$

[1] But see [39] for some important caveats.

The vacuum algebraic curve has poles at $x = \pm 1$, controlled by a vector κ_ℓ:

$$p_\ell = \frac{\kappa_\ell x}{x^2 - 1}, \qquad \kappa = \frac{\Delta}{2g}(0, -1, 0,\ 0, 1, 0)\,. \tag{6}$$

This must satisfy $S_{\ell m}\kappa_m = -\kappa_\ell$ and $\kappa_\ell A_{\ell m}\kappa_m = 0$; the particular solution is chosen by explicitly calculating the monodromy matrix [14] for the BMN point particle [43].

Solutions above this vacuum are constructed by introducing various cuts. The crucial equation here is that when crossing a cut C (of mode number n) in sheet ℓ, the change in p_ℓ is given by

$$p_\ell \to p_\ell - A_{\ell m} p_m + 2\pi n\,. \tag{7}$$

When $A_{\ell\ell} = 2$ this gives the change expected for a square root cut, but this is the more general form. As we approach the branch point, the change must go to zero, since continuity demands that it must agree with the result of walking around the end of the cut. This gives an equation for the positions of branch points: $2\pi n = A_{\ell m} p_m(x)$.

It will be useful to also write another set of quasimomenta q_i, corresponding to the basis directions in the following representation of the weight vectors:

$$\begin{array}{ccc|ccc|cc}
\Lambda_1 & \Lambda_2 & \Lambda_3 & \Lambda_{\bar 1} & \Lambda_{\bar 2} & \Lambda_{\bar 3} & & \\
\bigcirc & \oplus & \bigcirc & \bigcirc & \oplus & \bigcirc & & i \\
 & -1 & & & & & F & 1 \\
\hline
2\sin^2\phi & -1 & 2\cos^2\phi & & & & B & 3 \\
-\sin 2\phi & & \sin 2\phi & & & & B & 5 \\
 & & & & 1 & & F & 2 \\
\hline
 & & & -2\sin^2\phi & 1 & -2\cos^2\phi & B & 4 \\
 & & & \sin 2\phi & & -\sin 2\phi & B & 6
\end{array} \tag{8}$$

This is a solution to $\Lambda_\ell \cdot \Lambda_m = A_{\ell m}$ which reduces to the vectors [14] gave for $\alpha = \frac{1}{2}$ at least on the left (i.e for the unbarred $\ell = 1, 2, 3$). I have inserted a minus into the right half (barred ℓ) for later convenience. In terms of the $q_i(x)$, the algebraic curve with the vacuum plus resolvent G_1 turned on is:

$$\begin{pmatrix} q_1 \\ q_2 \\ \hline q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = \begin{pmatrix} \frac{\Delta}{2g}\frac{x}{x^2-1} \\ \frac{\Delta}{2g}\frac{x}{x^2-1} \\ \hline \frac{\Delta}{2g}\frac{x}{x^2-1} + 2\sin^2\phi\, G_1(x) + 2\cos^2\phi\, G_3(x) \\ \frac{\Delta}{2g}\frac{x}{x^2-1} - 2\sin^2\phi\, G_1(\frac{1}{x}) - 2\cos^2\phi\, G_3(\frac{1}{x}) \\ -\sin 2\phi\, G_1(x) + \sin 2\phi\, G_3(x) \\ +\sin 2\phi\, G_1(\frac{1}{x}) - \sin 2\phi\, G_3(\frac{1}{x}) \end{pmatrix} \to \frac{1}{2gx}\begin{pmatrix} \Delta + S \\ \Delta - S \\ \hline J' - Q' \\ J' + Q' \\ Q_5 \\ Q_6 \end{pmatrix} + \mathcal{O}\Big(\frac{1}{x^2}\Big). \tag{9}$$

This is reminiscent of $AdS_4 \times CP^3$ in that the bosonic resolvents G_1 and G_3 each appear on two sheets, one of them lacking the pole with Δ/g from the vacuum: we would call these light modes. $G_{\bar 1}$ and $G_{\bar 3}$ are similar. The terms $G(\frac{1}{x})$ have been filled in by the inversion symmetry, which now reads

$$q_2(x) = -q_1(\tfrac{1}{x}), \qquad q_4(x) = -q_3(\tfrac{1}{x}), \qquad q_6(x) = -q_5(\tfrac{1}{x}).$$

The global charges of the string are given by the large-x behaviour of the quasimomenta.

In general let us define J_ℓ corresponding to the Cartan generators, and $Q_i = J_\ell \Lambda_{\ell i}$:

$$p_\ell \to \frac{1}{2gx} J_\ell, \qquad q_i \to \frac{1}{2gx} Q_i \qquad \text{as } x \to \infty.$$

The right hand side of (9) defines charges Δ, S from the AdS directions, and J', Q' from the spheres. In terms of J_ℓ these are

$$\begin{aligned} \Delta &= \tfrac{1}{2}(-J_2 + J_{\bar 2}) \\ S &= \tfrac{1}{2}(-J_2 - J_{\bar 2}) \\ J' &= \sin^2\phi\,(J_1 - J_{\bar 1}) - \tfrac{1}{2}(J_2 - J_{\bar 2}) + \cos^2\phi\,(J_3 - J_{\bar 3}) \\ Q' &= -\sin^2\phi\,(J_1 + J_{\bar 1}) + \tfrac{1}{2}(J_2 + J_{\bar 2}) - \cos^2\phi\,(J_3 + J_{\bar 3}) \end{aligned} \tag{10}$$

and we will also want

$$\begin{aligned} J &= \ J_1 - J_{\bar 1} \ - J_2 + J_{\bar 2} \ + J_3 - J_{\bar 3} \\ Q &= -J_1 - J_{\bar 1} \ + J_2 + J_{\bar 2} \ - J_3 - J_{\bar 3}\,. \end{aligned} \tag{11}$$

For solutions with nonzero worldsheet momentum (i.e. solutions which are not by themselves closed strings) we must allow the quasimomentum to have a constant term at infinity:

$$p_\ell \to P_\ell + \frac{1}{2gx} J_\ell + \mathcal{O}\Big(\frac{1}{x}\Big).$$

The total momentum is given by

$$P = 2\sin^2\phi\,\left(-P_1 + P_{\bar 1}\right) + 2\cos^2\phi\,\left(-P_3 + P_{\bar 3}\right). \tag{12}$$

Constructing Modes

The first fluctuation of the vacuum solution is given by turning on a new pole with canonical residue $-\alpha(y)$ [44]:

$$G_1(x) = -\frac{\alpha(y)}{x-y} + \frac{1}{2}\frac{\alpha(y)}{-y}, \qquad \alpha(y) = \frac{1}{2g}\frac{y^2}{y^2-1}.$$

The perturbation may also alter the residues at ± 1, and at infinity it must behave as follows:

$$\begin{pmatrix} \delta q_1 \\ \delta q_2 \\ \hline \delta q_3 \\ \delta q_4 \\ \delta q_5 \\ \delta q_6 \end{pmatrix} = \begin{pmatrix} \delta K \\ \delta K \\ \hline \delta K + 2\sin^2\phi\,[\frac{\alpha(y)}{x-y} + \frac{\alpha(y)}{2y}] \\ \delta K - 2\sin^2\phi\,[\frac{\alpha(y)}{1/x-y} + \frac{\alpha(y)}{2y}] \\ \delta K_5 - \sin 2\phi\,[\frac{\alpha(y)}{x-y} + \frac{\alpha(y)}{2y}] \\ \delta K_6 + \sin 2\phi\,[\frac{\alpha(y)}{1/x-y} + \frac{\alpha(y)}{2y}] \end{pmatrix} \to \frac{1}{2gx}\begin{pmatrix} \delta\Delta \\ \delta\Delta \\ \hline 2\sin^2\phi \\ 0 \\ -\sin 2\phi \\ 0 \end{pmatrix} + \mathcal{O}\Big(\frac{1}{x^2}\Big).$$

Here $\delta K = \frac{\delta\Delta}{2g}\frac{x}{x^2-1}$ for the first four sheets, synchronised as in (6), and $\delta K_5 = \delta K_6 = b\frac{x}{x^2-1}$.[2] From the sheets connected by the new poles in $\delta q(x)$ we might call this the $(3,5)$ mode. Solving the conditions at infinity, we get off-shell frequency

$$\delta\Delta = \Omega_1(y) = \frac{2\sin^2\phi}{y^2-1}. \tag{13}$$

[2] The pole $\delta K_5 = \delta K_6$ corresponds to $\kappa' = \frac{1}{2}b(-\cot\phi, 0, \tan\phi, \cot\phi, 0, -\tan\phi)$ which like (6) is a -1 eigenvector of S. The solution has $b = \frac{\sin 2\phi}{2g}\frac{1}{1-y^2}$.

The perturbation carries some momentum,[3] and solving the inversion conditions gives $P_1 = -P_{\bar{1}} \neq 0$ and

$$\delta P = \frac{\sin^2\phi}{g} \frac{y}{y^2-1}. \tag{14}$$

Now to find the position of this mode in the spectral plane, we take (7) and demand continuity as we approach the end of the infinitesimally short branch cut which we are inserting. This gives[4]

$$\begin{aligned} 2\pi n_1 &= A_{1k} p_k = 4\sin^2\phi\, p_1 - 2\sin^2\phi\, p_2 \\ &= 2\sin^2\phi \frac{\Delta}{2g} \frac{x}{x^2-1} \end{aligned} \tag{15}$$

using the vacuum solution (6) on the second line. Solving for x, and choosing the solution outside the unit circle, we get

$$x_n = \frac{\Delta \sin^2\phi}{4\pi\, g\, n} \pm \sqrt{1 + \Big(\frac{\Delta\sin^2\phi}{4\pi\, g\, n}\Big)^2}$$

giving the desired on-shell frequency

$$\omega_n = \Omega_1(x_n) = -\sin^2\phi + \sqrt{\sin^4\phi + \Big(\frac{4\pi\, g\, n}{\Delta}\Big)^2}. \tag{16}$$

The charges (10), (11) are

$$\delta J = -1, \qquad \delta Q = +1, \qquad \delta J' = -\sin^2\phi, \qquad \delta Q' = \sin^2\phi$$

and the momentum (14) is $\delta P = 2\pi n/\Delta$, so that if $\omega = \Delta - J' - Q'$ this matches (2).

The construction of all the other modes is similar:

$1f$. A fermion of the same mass is obtained by turning on $G_1(x) = G_2(X) = -\frac{\alpha(y)}{x-y} + \frac{1}{2}\frac{\alpha(y)}{-y}$. The equation for the perturbation can be written

$$\begin{pmatrix} \delta q_1 \\ \hline \delta q_3 \\ \delta q_5 \end{pmatrix} = \begin{pmatrix} \delta K + [\frac{\alpha(y)}{x-y} + \frac{\alpha(y)}{2y}] \\ \hline \delta K + (1-2\sin^2\phi)\,[\frac{\alpha(y)}{x-y} + \frac{\alpha(y)}{2y}] \\ \delta K_5 + \sin 2\phi\,[\frac{\alpha(y)}{x-y} + \frac{\alpha(y)}{2y}] \end{pmatrix} \to \frac{1}{2gx} \begin{pmatrix} \delta\Delta + 1 \\ \hline 1 - 2\sin^2\phi \\ \sin 2\phi \end{pmatrix} + \ldots$$

(with the rest filled in by inversion symmetry as before). At $\phi = \frac{\pi}{4}$ this has new poles on just two sheets and thus might be called the $(1,5)$ mode, but for general ϕ this interpretation is not clear; let us call it "$1f$". The resulting off-shell frequency is the same, $\Omega_{1f}(y) = \Omega_1(y)$. The positions of the poles come from

$$\begin{aligned} 2\pi n_{1f} &= \sum_{\ell=1,2} A_{\ell k} p_k \\ &= 2\sin^2\phi\, p_1 - 2\sin^2\phi\, p_2 - 2\cos^2\phi\, p_3 \\ &= 2\sin^2\phi \frac{\Delta}{2g}\frac{x}{x^2-1} \qquad \text{(for the vacuum).} \end{aligned} \tag{17}$$

Thus we will get exactly the same frequencies ω_n as for the boson.

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[3] This momentum is often avoided by considering a pair of fluctuations at $\pm y$ as in [45, 27], see also [28]. Doing so for the giant magnon, you would miss the second term in (24).

[4] Compared to $q_3 - q_5 = (2\sin^2\phi + \sin 2\phi)p_1 - p_2 + (2\cos^2\phi - \sin 2\phi)$ we see that (15) reduces to $2\pi n = q_3 - q_5$ as in [40] only at $\phi = \frac{\pi}{4}$. The same is true for the other modes.

r	m_r		$2\pi n_r$	δJ	δQ	$\alpha = \frac{1}{2}$ i.e. $\phi = \frac{\pi}{4}$
$1, \bar{1}$	$\sin^2\phi$	●—⊕—○	$A_{1k}p_k$, or $-A_{\bar{1}k}p_k$	-1	$1, -1$	$(3,5), (4,6)$
$1f, \bar{1}f$	$\sin^2\phi$	●—⊕—○	$(A_{1k}+A_{2k})p_k$, or sim.	0	0	$(1,5), (2,6)$
$3, \bar{3}$	$\cos^2\phi$	○—⊕—●	$A_{3k}p_k$	-1	$1, -1$	$(3,-5), (4,-6)$
$3f, \bar{3}f$	$\cos^2\phi$	○—⊕—●	$(A_{2k}+A_{3k})p_k$	0	0	$(1,-5), (2,-6)$
$4, \bar{4}$	1	●—⊕—●	$(A_{1k}+2A_{2k}+A_{3k})p_k$	0	0	$(1,-1), (2,-2)$
$4f, \bar{4}f$	1	●—⊕—●	$(A_{1k}+A_{2k}+A_{3k})p_k$	-1	$1, -1$	$(1,-3), (2,-4)$

Table 1: List of modes in the $AdS_3 \times S^3 \times S^3$ algebraic curve. The colouring of the nodes is $-k_{\ell r}$ with ● $= +1, -1$ and ● $= +2, -2$, writing "left, right" everywhere.

3. We can treat the boson "3" (from $G_3 = -\frac{\alpha(y)}{x-y} + \frac{1}{2}\frac{\alpha(y)}{-y}$) and the fermion "$3f$" in exactly the same way, obtaining

$$\Omega_3(y) = \Omega_{3f}(y) = \frac{2\cos^2\phi}{y^2-1}$$
$$\omega_n = -\cos^2\phi + \sqrt{\cos^4\phi + \Big(\frac{4\pi\, g\, n}{\Delta}\Big)^2}.$$

Note that δq_5 has the opposite sign for these modes compared to $1, 1f$ above; at $\phi = \frac{\pi}{4}$ we might therefore call them $(3,-5)$ and $(1,-5)$, thinking of $q_{-i} = -q_i$ as another six sheets.

4. The heavy modes can be constructed by simply adding two light modes: $4 = 1f + 3f$, and $4f = 1 + 3f = 3 + 1f$. This addition is at the level of $\delta q_i(x)$, and hence applies to $\Omega_r(y)$ too:

$$\Omega_4(y) = \Omega_{4f}(y) = \frac{2}{y^2-1}.$$

For the mode numbers, we have[5]

$$\begin{aligned} 2\pi n_4 &= (A_{1k}+2A_{2k}+A_{3k})p_k = 2\pi n_{1f} + 2\pi n_{3f} \\ &= -2p_2 = 2\frac{\Delta}{2g}\frac{x}{x^2-1} \end{aligned} \tag{18}$$

and thus

$$\omega_n = -1 + \sqrt{1 + \Big(\frac{4\pi\, g\, n}{\Delta}\Big)^2}.$$

For the heavy boson we must add two fermions, not two bosons: $4 \neq 1+3$. This and $\bar{4}$ are the two transverse direction in AdS_3; unlike the CP^3 case there are no heavy bosons in the sphere directions.

$\bar{1}$. Finally, the barred modes differ only by some minus signs: $G_{\bar{1}}(x) = +\frac{\alpha(y)}{x-y} - \frac{1}{2}\frac{\alpha(y)}{-y}$ and $2\pi n_{\bar{1}} = -A_{\bar{1}k}p_k$, and $\delta Q = -1$. But (13) and (16) are the same.

The addition of modes used above is one half of [42]'s efficient procedure for constructing off-shell frequencies, and these barred modes may be constructed by the other half, namely the use of the inversion symmetry. Writing the perturbation for mode r (with a pole at y) as $\delta_r^{(y)} p_\ell(x)$, the new pole for the corresponding mode on the right is given

[5] As in $AdS_4 \times CP^3$ the heavy modes correspond to stacks of Bethe roots [46]. In general few solutions x here will make n_{1f} and n_{3f} integers, but in the thermodynamic limit $\Delta/g \to \infty$ this constraint disappears.

by $\delta_{\bar{r}}^{(y)} p_\ell(x) = -\delta_r^{(1/y)} p_\ell(x)$, exactly as in [42].

Table 1 summarises some properties of these modes. Another summary is as follows: The perturbation δq_i for N_r excitations of the mode r has a pole at y with residue $k_{ir} N_r \alpha(y)$, where k_{ir} are the coefficients inside the bracket below. This condition at infinity constrains the perturbation sufficiently that we can read off $\Omega(y) = \delta\Delta$:

$$\begin{pmatrix} \delta q_1 \\ \delta q_2 \\ \hline \delta q_3 \\ \delta q_4 \\ \delta q_5 \\ \delta q_6 \end{pmatrix} \to \frac{1}{2gx} \begin{pmatrix} \delta\Delta + 2N_4 + N_{1f} + N_{3f} + N_{4f} \\ \delta\Delta + 2N_{\bar{4}} + N_{\bar{1}f} + N_{\bar{3}f} + N_{\bar{4}f} \\ \hline -2\sin^2\phi\,(N_1 + N_{1f}) + N_{1f} - 2\cos^2\phi\,(N_3 + N_{3f}) + N_{3f} - N_{4f} \\ -2\sin^2\phi\,(N_{\bar{1}} + N_{\bar{1}f}) + N_{\bar{1}f} - 2\cos^2\phi\,(N_{\bar{3}} + N_{\bar{3}f}) + N_{\bar{3}f} - N_{\bar{4}f} \\ \sin 2\phi\,(N_1 + N_{1f} - N_3 - N_{3f}) \\ \sin 2\phi\,(N_{\bar{1}} + N_{\bar{1}f} - N_{\bar{3}} - N_{\bar{3}f}) \end{pmatrix} + \ldots \tag{19}$$

The analogous equation in terms of p_ℓ has $\delta p_\ell \to \frac{1}{2gx}(-\frac{1}{2}, -1, -\frac{1}{2},\ \frac{1}{2}, 1, \frac{1}{2})\delta\Delta + \frac{1}{2gx} k_{\ell r} N_r$ as $x \to \infty$, with $k_{\ell r} = \pm 1, \pm 2$ taken from the colouring in table 1.

3. Summation Prescriptions

Semiclassical quantisation gives the one-loop correction to the energy of a soliton as

$$\delta E = \sum_r \sum_{n=-\infty}^{\infty} (-1)^{F_r} \frac{1}{2} \omega_n^r - \delta E_{\text{vac}}\,. \tag{20}$$

For us $\delta E_{\text{vac}} = 0$. The sum for any one polarisation r will diverge quadratically, but for a matched set of bosons and fermions cancellations typically tame this to a logarithmic divergence. This still leaves some room for dependence on how we cut off the sum on n in the UV.

In terms of the spectral plane, a very high-energy mode is one located very close to $x = 1$, with energy

$$\omega = \Omega_r(1+\epsilon) = \frac{2m_r}{\epsilon(2+\epsilon)} = \frac{m_r}{\epsilon} + \mathcal{O}(\epsilon^0).$$

In terms of mode numbers, instead

$$\omega_N = -m_r + \sqrt{m_r^2 + \Big(\frac{4\pi g N}{\Delta}\Big)^2} = N\frac{4\pi g}{\Delta} - m_r + \mathcal{O}\Big(\frac{1}{N}\Big).$$

The principal options for how to regulate the modes of different masses follow from these:

i. A cutoff at a fixed physical energy $\omega = \Lambda$ is a cutoff at the same mode number for all polarisations (provided there are no divergences stronger than $\log N$), but at radius $x = 1 + m_r/\Lambda$ in the spectral plane for a mode of mass m_r. It is what would seem most natural to a resident of the target space ignorant of integrability, and is sometimes referred to as the worldsheet prescription, although (as we showed in [28]) can quite easily be implemented in the algebraic curve description.[6]

ii. The alternative is a cutoff at fixed radius in the spectral plane, which corresponds to a cutoff at mode number $N_r = m_r \frac{\Delta}{4\pi g}\frac{1}{\epsilon}$. This is certainly the easiest to implement in the algebraic curve language, although it can clearly also be used to add up frequencies from a worldsheet calculation, as Gromov and Mikhaylov [25] did upon introducing this

[6] Observe also that this "physical" prescription corresponds also to a cutoff in worldsheet momentum, to the precision required here. At $x = 1+\epsilon$ we have from (14) and the equivalent for the m_3 modes $P \propto \omega + \mathcal{O}(\epsilon^0)$.

idea for $AdS_4 \times CP^3$. To do so however we still need to identify the heavy modes, which for classical solutions far from the BMN vacuum is not necessarily obvious. For this the algebraic curve is definitive: the heavy modes are those whose off-shell perturbation is the sum of two light modes'.

In the present $AdS_3 \times S^3 \times S^3 \times S^1$ case this "new" prescription leads to three different cutoffs in terms of energy, or mode number, and these clearly change as $\alpha = \cos^2 \phi$ is changed. It will be important below that as we approach $\alpha = 1$, where one of the spheres decompactifies, the cutoff for the mode which is becoming massless drops to zero, completely excluding this mode from the sum. (See also the integral form (28) below.)

One argument advanced in favour of the new prescription in $AdS_4 \times CP^3$ involves the fact that the total energy of a pair of light modes exactly at their cutoffs is the same as that of the corresponding heavy mode at its cutoff [29]. This is still true here as each heavy mode is made of two light modes of opposite mass, and $m_1 + m_3 = m_4$. There is a third option which also has this property:

iii. We could simply cut off both light modes at half the energy of the heavy mode. In terms of mode numbers and the spectral plane this means

$$N_1 = N_3 = \tfrac{1}{2} N_4, \qquad \begin{aligned} \epsilon_1 &= 2 m_1 \epsilon_4 \\ \epsilon_3 &= 2 m_3 \epsilon_4 \,. \end{aligned} \tag{21}$$

For $\alpha = \frac{1}{2}$ (and for AdS_4) this is identical to the new prescription. It avoids turning off the newly massless modes m_1 as we approach $\alpha = 1$, but once we get there still treats the modes $m_3 = 1$ and $m_4 = 1$ differently. For this reason it seems undesirable, and I will mention it only as an afterthought.

4. Giant Magnons

Giant magnons are the macroscopic classical string solutions corresponding to elementary excitations with momentum p of order 1 [30]. Bound states of a large number $Q \sim g$ of magnons form dyonic giant magnons [47]; from the algebraic point of view this is the natural case, and they are described by a single log cut [48]. The branch points of this are the Zhukovsky variables $X^{\pm}$, which are defined here by [15]

$$X^+ + \frac{1}{X^+} - X^- - \frac{1}{X^-} = i \frac{2 m_r}{h} Q, \qquad \frac{X^+}{X^-} = e^{ip}$$

with $Q = 1$ for an elementary excitation. The exact dispersion relation is

$$E(p) = -i \frac{h}{2} \left(X^+ - \frac{1}{X^+} - \text{c.c.} \right) = \sqrt{Q^2 m_r^2 + 4 h^2 \sin^2 \frac{p}{2}}. \tag{22}$$

At strong coupling we can expand this using $h = 2g + c + \mathcal{O}(1/g)$ to get the classical energy and one-loop correction $E_0 + \delta E + \ldots$. This is to be done holding p and Q fixed.

The classical solution for a magnon on the first sphere is (9) with

$$G_1(x) = \frac{1}{2 \sin^2 \phi} \Big[G_{\text{mag}}(x) - \tfrac{1}{2} G_{\text{mag}}(0) \Big], \qquad G_{\text{mag}}(x) = -i \log \left(\frac{x - X^+}{x - X^-} \right). \tag{23}$$

The prefactor is needed to cancel the non-integer factor in (15), from (7), and also in (12).

The asymptotic charges (10), (11) for this are

$$J' = \Delta + ig\left(X^+ - \frac{1}{X^+} - \text{c.c.}\right)$$
$$Q' = \sin^2\phi\, Q = -ig\left(X^+ + \frac{1}{X^+} - \text{c.c.}\right)$$

and the momentum is

$$P = G_{\text{mag}}(0) = p.$$

This clearly gives precisely the desired dispersion relation with $E_0 = \Delta - J'$.

There is of course a similar magnon on the second sphere, with $G_3 = \frac{1}{2\cos^2\phi}[G_{\text{mag}}(x) - \frac{p}{2}]$, a giant version of the mode "3". Note however that there is no analogue of the RP^3 giant magnon in $AdS_4 \times CP^3$, in which turning on magnons in two sectors led to a simplification. That solution was was a giant version of the CP^3 heavy boson; here the only heavy bosons are the AdS modes 4 and $\bar{4}$. (See however footnote 7 in the conclusions.)

One-loop Correction

To calculate the one-loop correction to the energy we should begin by finding the off-shell frequencies, by constructing the perturbations of the quasimomenta. The two differences from the BMN modes above are that here is that we must allow the endpoints of the cut to move, and that we do not allow the perturbation to alter the total momentum. For instance for the $3f$ mode and the magnon (23), the perturbation must obey

$$\delta q = \left(\begin{array}{l} \delta K \\ \delta K \\ \hline \delta K + (1-2\cos^2\phi)\,[\frac{\alpha(y)}{x-y} + \frac{\alpha(y)}{2y}] + 2\sin^2\phi\,[H(x) - \frac{1}{2}H(0)] \\ \delta K + (1-2\cos^2\phi)\,[\frac{\alpha(y)}{1/x-y} + \frac{\alpha(y)}{2y}] - 2\sin^2\phi\,[H(\frac{1}{x}) - \frac{1}{2}H(0)] \\ \delta K_5 - \sin 2\phi\,[\frac{\alpha(y)}{x-y} + \frac{\alpha(y)}{2y}] - \sin 2\phi\,[H(x) - \frac{1}{2}H(0)] \\ \delta K_6 - \sin 2\phi\,[\frac{\alpha(y)}{1/x-y} + \frac{\alpha(y)}{2y}] + \sin 2\phi\,[H(\frac{1}{x}) - \frac{1}{2}H(0)] \end{array}\right) \to \frac{1}{2gx}\left(\begin{array}{l} \delta\Delta \\ \delta\Delta \\ \hline 1-2\cos^2\phi \\ 0 \\ -\sin 2\phi \\ 0 \end{array}\right) + \ldots$$

where $H(x) = \frac{A_+}{x-X^+} + \frac{A_-}{x-X^-}$ has been inserted wherever the classical solution has $G_{\text{mag}}(x)$. The resulting frequency is the $r = 3f$ case of

$$\Omega_r(y) = \frac{2\,m_r}{y^2-1}\left(1 - y\frac{X^+ + X^-}{X^+X^- + 1}\right) = m_r\,\Omega_{\text{mag}}(y). \tag{24}$$

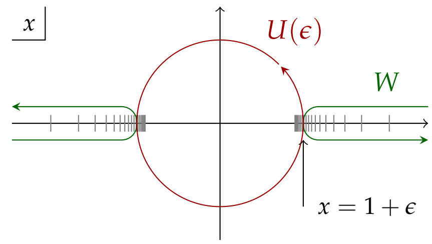
The same formula applies to all light and heavy modes, and equally for the magnon in G_3. In fact it is the same as in S^5 and CP^3, and as discussed in [49] the first term can be interpreted as the energy of the mode itself (13), and the second as the effect of adjusting the magnon's momentum to compensate for the momentum (14) carried by the perturbation.

While we cannot write the frequencies ω_n in closed form, we can still compute the one-loop correction. This is done by using [50]'s trick to write it as a contour integral in n:

$$\delta E = \sum_{n,r} \frac{1}{2}\omega_n^r = \frac{1}{4i}\oint dn\,\sum_r (-1)^{F_r}\cot(\pi n)\,\Omega_r(x_n^r).$$

For each r we can now use the relevant equation for the position of the pole to write this as an integral in x, along a contour W enclosing poles along the real line at $|x| > 1$. Then we

deform this contour to the unit circle, $-U$ counting orientation:



Here we can approximate $\cot(\pi n) \approx \cot(m_r \frac{\Delta}{g} \frac{x}{x^2-1}) \approx \pm i$ on the upper/lower semicircles $U_\pm$, whose contributions are equal, to write:

$$\begin{aligned} \delta E &= \frac{1}{4i} \oint_W dx \sum_r (-1)^{F_r} \cot(\pi n_r)\, \partial_x n_r(x)\, \Omega_r(x) \\ &\approx \frac{-1}{4\pi} \int_{U_+} \sum_r (-1)^{F_r} 2\pi \partial_x n_r(x)\, \Omega_r(x). \end{aligned} \tag{25}$$

Since $\Omega_r = m_r \Omega_{\text{mag}}$, let us deal with n_r by considering the sum over polarisations one mass at a time. For $m_r = \sin^2\phi$ we see

$$\begin{aligned} \sum_{r=1,1f,\bar{1},\bar{1}f} (-1)^{F_r} 2\pi n_r &= 2\sin^2\phi\, p_1 + 2\cos^2\phi\, p_3 - 2\sin^2\phi\, p_{\bar{1}} - 2\cos^2\phi\, p_{\bar{3}} \\ &= G_{\text{mag}}(x) - G_{\text{mag}}(\tfrac{1}{x}). \end{aligned}$$

For the other masses, from table 1 it is clear that this sum for the $m_r = \cos^2\phi$ modes is identical, while that for $m_r = 1$ is exactly minus this. Finally we should allow a different cutoff for each mass, giving us the same log-divergent integral thrice:

$$\begin{aligned} \delta E &= \frac{-1}{4\pi} \left\{ \sin^2\phi \int_{U_+(\epsilon_1)} dx + \cos^2\phi \int_{U_+(\epsilon_3)} dx - \int_{U_+(\epsilon_4)} dx \right\} \partial_x \left[G_{\text{mag}}(x) - G_{\text{mag}}(\tfrac{1}{x}) \right] \Omega_{\text{mag}}(x) \\ &= \frac{i}{\pi} \frac{X^+ - X^-}{X^+ X^- + 1} \left[\sin^2\phi\, \log\epsilon_1 + \cos^2\phi\, \log\epsilon_3 - \log\epsilon_4 \right]. \end{aligned}$$

Clearly this will vanish for the new prescription, with all ϵ_r the same. For the physical prescription $\epsilon_r = m_r/\Lambda$ and (taking the non-dyonic limit $X^\pm = e^{\pm ip/2}$) instead

$$\delta E = \frac{1}{\pi} \sin\frac{p}{2} \left[\sin^2\phi\, \log(\sin^2\phi) + \cos^2\phi\, \log(\cos^2\phi) \right].$$

Since $\delta E = 2\,c\,\sin\frac{p}{2}$ this implies $c_{\text{new}} = 0$ and

$$\begin{aligned} c_{\text{phys}} &= \frac{\sin^2\phi\, \log(\sin^2\phi) + \cos^2\phi\, \log(\cos^2\phi)}{2\pi} \\ &= \begin{cases} -\frac{\log 2}{2\pi} & \phi = \frac{\pi}{4} \\ 0 & \phi = 0. \end{cases} \end{aligned} \tag{26}$$

Using the third prescription, (21), gives $c_{\text{third}} = c_{\text{phys}} + \frac{\log 2}{2\pi}$; as expected this matches c_{new} at $\phi = \frac{\pi}{4}$.

For the dyonic case the calculation works equally well, provided we expand the dispersion relation holding fixed not $X^\pm$ but rather p and Q [28]:

$$\delta E = c \frac{\partial E_0}{\partial h}\Big|_{Q,p} = -2i \frac{X^+ - X^-}{X^+ X^- + 1} c\,. \tag{27}$$

	"BMN" $\nu \to \kappa$:	"AdS" $\nu \to 0$:	r, $\bar{r}$
Bosons:			
$\omega^B_{1,2} = \lvert n \rvert$	$\to \lvert n \rvert$	$\to \lvert n \rvert$	$\times$
$\omega^B_{3,4} = \sqrt{n^2 + \alpha^2 \nu^2}$	$\to \sqrt{n^2 + \alpha^2 \kappa^2}$	$\to \lvert n \rvert$	3
$\omega^B_{5,6} = \sqrt{n^2 + (1-\alpha)^2 \nu^2}$	$\to \sqrt{n^2 + (1-\alpha)^2 \kappa^2}$	$\to \lvert n \rvert$	1
$\omega^B_{7,8} = \sqrt{n^2 + 2\kappa^2 \mp 2\sqrt{n^2\nu^2 + \kappa^4}}$	$\to \pm\kappa^2 + \sqrt{n^2 + \kappa^2}$	$\to \begin{cases} \lvert n \rvert \\ \sqrt{n^2 + 4\kappa^2} \end{cases}$	4
Fermions:			
$\omega^F_{1,2} = \pm\frac{\nu}{2} + \lvert n \rvert$	$\to \pm\frac{\kappa}{2} + \lvert n \rvert$	$\to \lvert n \rvert$	$\times$
$\omega^F_{5,6}\rvert_{\alpha=1/2} = \sqrt{n^2 + \frac{\kappa^2}{2} + \sqrt{n^2\nu^2 + \frac{\kappa^4}{4}}}$	$\to \frac{\kappa}{2} + \sqrt{n^2 + \frac{\kappa^2}{4}}$	$\to \sqrt{n^2 + \kappa^2}$	$1f, 3f$
$\omega^F_{7,8}\rvert_{\alpha=1/2} = \sqrt{n^2 + \frac{\kappa^2}{2} - \sqrt{n^2\nu^2 + \frac{\kappa^4}{4}}}$	$\to -\frac{\kappa}{2} + \sqrt{n^2 + \frac{\kappa^2}{4}}$	$\to \lvert n \rvert$	$1f, 3f$
$\omega^F_{3,4} = \pm\frac{\nu}{2} + \sqrt{n^2 + \kappa^2}$	$\to \pm\frac{\kappa}{2} + \sqrt{n^2 + \kappa^2}$	$\to \sqrt{n^2 + \kappa^2}$	$4f$

Table 2: Modes of the folded spinning string in $AdS_3 \times S^3 \times S^3 \times S^1$, taken from [16]'s appendix C.

5. Long Spinning Strings

The aim of this section is to do what Gromov and Mikhaylov [25] did for $AdS_4 \times CP^3$, that is, to classify the modes of long spinning strings as heavy or light based on the algebraic curve description, and then to sum their frequencies using the "new" prescription suggested by this formalism.

As reviewed in section 3, the new prescription cuts off at the same radius $1 + \epsilon$ in the spectral plane for all modes, which corresponds mode number $N_r = m_r N$ for modes of mass m_r. Given the explicit frequencies ω_n (and since $\kappa \gg 1$) we can write the sum as the following integral:

$$\delta\Delta_{\text{new}} = \sum_r (-1)^{F_r} \int_{-m_r N}^{m_r N} dn \, \frac{1}{2}\omega^r_n = \int_{-N}^{N} dn \, \sum_r (-1)^{F_r} \frac{1}{2} m_r \omega^r_{m_r n} \,. \tag{28}$$

The classical solution of interest is a folded string spinning in AdS_3, stretched all the way to the boundary, times a point particle with some momentum along the equators of both spheres. (See (43) in appendix C.) For this solution Forini, Puletti and Ohlsson Sax [16] calculated the mode frequencies from the Green–Schwartz action; these are listed in table 2. Some comments:

- The parameter $\nu = J'/4\pi g$ controls the momentum on the spheres, and we are interested in two limits. First, setting $\nu = \kappa$ takes us to the BMN limit, and it is here that the mass m_r relevant for classifying the modes is literally the mass of the excitation, as in (16). Second, sending $\nu \to 0$ gives us the simplest rotating string entirely in AdS_3, whose frequencies should be independent of α.

- In the first two columns only $\omega^F_{5,6}$ and $\omega^F_{7,8}$ assume $\alpha = \frac{1}{2}$. We know that the light fermions 5, 6 cannot both be $r = 1f$ (or both $r = 3f$) as this would break the symmetry between the two spheres; this explains the classification in the last column.

It is now very simple to put these modes into an integral like that above. Allowing three

different cutoffs, the result is:

$$\begin{aligned}\delta\Delta &= \sum_{m_r=\sin^2\phi}(-1)^{F_r}\sum^{N_1}\tfrac{1}{2}\omega_n^r + \sum_{m_r=\cos^2\phi}(-1)^{F_r}\sum^{N_3}\tfrac{1}{2}\omega_n^r + \sum_{m_r=1}(-1)^{F_r}\sum^{N_4}\tfrac{1}{2}\omega_n^r \\ &= \kappa\left[-\log 4 + \log N_4 - \tfrac{1}{2}\log N_3 - \tfrac{1}{2}\log N_1\right] \\ &= \begin{cases} -\kappa\log 4 & \text{using the physical prescription} \\ -\kappa\log(2\sin 2\phi) & \text{new} \\ -\kappa\log 2 & \text{third}\,. \end{cases}\end{aligned} \tag{29}$$

where $\kappa = \frac{1}{\pi}\log S$. The result for the physical prescription is precisely that given in [16].

The result for the new sum prescription is not what we would expect based on the giant magnon results, which is this:

$$\delta\Delta = \delta\Delta_{\text{phys}} + (c_{\text{new}} - c_{\text{phys}})\frac{\partial\Delta_0}{\partial h} = -\kappa\log 2$$

writing $\Delta_0 = 2h\log S$ for the leading term in (3). Note also that there is a divergence at $\phi = 0$ i.e. at $\alpha = 1$. The reason for this is that ω_5^B and ω_5^F are declared massless in this limit, and hence omitted from (28). Unlike the unambiguously massless modes $\omega_{1,2}^B$ and $\omega_{1,2}^F$ which clearly always cancel out, these do not. In fact the only difference between the two sums (at $\alpha = 1$) is that the new prescription omits these two modes' contribution:

$$\sum_{n=-N_1}^{N_1}\omega_5^B - \omega_5^F = -\log N_1 - \tfrac{1}{2} - \log 2 + \mathcal{O}\Big(\frac{1}{N_1}\Big). \tag{30}$$

Consequences

In order to use these results to find c, we need to know the subleading term in the expansion in h, i.e. f_1 in (3):

$$\delta\Delta = (2c + f_1)\kappa\pi.$$

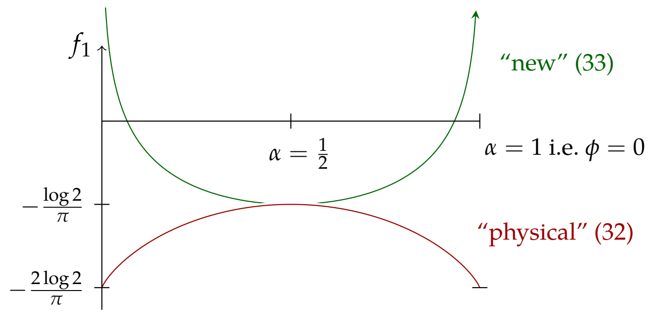
It was noted in [16] that the Bethe equations at $\alpha = 1$ for the relevant $sl(2)$ sector are identical to those for $AdS_5 \times S^5$, and if one then assumes that the dressing phase is also identical, then $f_1 = -\frac{3\log 2}{\pi}$ as in [35]. This led them to $c = \frac{\log 2}{4\pi}$. Likewise at $\alpha = \frac{1}{2}$ the equations are the same as for $AdS_4 \times CP^3$, where the dressing phase was identical, and $f_1 = -\frac{3\log 2}{2\pi}$ [21]; carrying this over to the present AdS_3 gave $c = -\frac{\log 2}{4\pi}$. We can now attempt a similar comparison for the new sum, where $\alpha = \frac{1}{2}$ (and still $f_1 = -\frac{3\log 2}{2\pi}$) leads to $c = +\frac{\log 2}{2\pi}$.

These values for c are in all cases different to those from the giant magnon, (26). It seems likely that what we are learning here is that the dressing phase is *not* the same as it was for $AdS_5 \times S^5$ and $AdS_4 \times CP^3$.

In this case we should instead use c from the giant magnon and $\delta\Delta$ from the spinning string to predict what f_1 will be. The answers are

$$f_1 = \begin{cases} -\frac{1}{\pi}\left[\log 4 + \sin^2\phi\log(\sin^2\phi) + \cos^2\phi\log(\cos^2\phi)\right] & \text{physical prescription} \\ -\frac{1}{\pi}\left[\log 4 + \log(2\sin 2\phi)\right] & \text{new prescription.} \end{cases}$$

These two agree at $\phi = \frac{\pi}{4}$, but away from this they disagree. Here is a graph:



This difference is quite unlike anything seen in the $AdS_4 \times CP^3$ case: The one-loop (i.e. $1/g$) corrections to the spinning string obtained using these two prescriptions cannot both follow from the same function of h. Thus we must conclude that at least one of them is incorrect. Given the divergence seen above at $\phi = 0$, it seems reasonable to say that it is the new prescription which is at fault.

Finally note that for the third cutoff (21), results both for the magnon and for the spinning string differ from those for the physical cutoff only by a $\log 2$ term. These cancel out to give exactly the same prediction for f_1 i.e. (32).

6. Comments and Conclusions

The first paper to write down the subleading term of the interpolating function $h(\lambda)$ for $AdS_3 \times S^3 \times S^3 \times S^1$ was [16], who gave

$$c = \begin{cases} -\frac{\log 2}{4\pi} & \alpha = \frac{1}{2} \\ \frac{\log 2}{4\pi} & \alpha = 1 \end{cases}, \qquad \text{assuming} \qquad f_1 = \begin{cases} -\frac{3\log 2}{2\pi} & \text{as for } CP^3 \\ -\frac{3\log 2}{\pi} & \text{as for } S^5. \end{cases}$$

These values for f_1 come from [16]'s observation that the Bethe equations for this $sl(2)$ sector are the same as those for $AdS_4 \times CP^3$ or $AdS_5 \times S^5$ at these two values of α, and then tentatively assuming that the dressing phase is also the same as that for both previous correspondences. As they note, there is no particular reason to think that this is true, and comparison with the giant magnon results now shows it not to be so. For the same value of α, the magnons give

$$c = \begin{cases} -\frac{\log 2}{2\pi} & \alpha = \frac{1}{2} \\ 0 & \alpha = 1. \end{cases} \tag{31}$$

Turning this comparison around we can instead use the magnon calculation (26) to predict the one-loop correction which should arise from the correct dressing phase, for all α. This gives:

$$f_1 = -\frac{1}{\pi} \Big[2\log 2 + \alpha \log \alpha + (1-\alpha)\log(1-\alpha) \Big] . \tag{32}$$

The giant magnon calculation here uses the algebraic curve and thus omits massless modes. This is a possible source of error, but see points 7, 8 below for evidence against this. The spinning string calculation includes the massless modes, but they cancel among themselves.

This result is for the "physical" summation prescription, i.e. using a cutoff at the same energy (or mode number) for all modes. One can instead consider the "new" prescription, defined with a cutoff at fixed radius in the spectral plane. This is one possible generalisation of the cutoff introduced by [25] for $AdS_4 \times CP^3$, but the story is a little different here:

1. If this prescription is equally valid, then adopting it should affect all one-loop results in the same way, namely $\delta E_{\text{new}} - \delta E_{\text{phys}} = (c_{\text{new}} - c_{\text{phys}})\partial E_0/\partial h\big|_{h=2g}$. However this is not what happens. One way to say this is to use $c = 0$ from the giant magnon and δE from the spinning string to predict f_1. This gives

$$f_1 = -\frac{1}{\pi}\left[2\log 2 + \tfrac{1}{2}\log\alpha + \tfrac{1}{2}\log(1-\alpha)\right] \tag{33}$$

disagreeing with (32) at $\alpha \neq \frac{1}{2}$. However f_1 is part of an expansion in h which should be independent of the prescription.

2. That there is something wrong with the new prescription is most clear at $\alpha = 1$. Here one of the modes it deems massless in the limit in fact plays an important role (for the long spinning string), and the effect of changing to the new prescription is simply to remove this from the sum, (30), producing a divergence. We would very much like to have a smooth $\alpha \to 1$ limit.

3. Instead of the new prescription we can consider a third prescription, (21), cutting off both kinds of light modes at half the energy of the heavy modes. At $\alpha = \frac{1}{2}$, and in $AdS_4 \times CP^3$, this is identical. In some way it may be closer to the spirit of [25], in that it treats heavy modes with mode number $2n$ alongside light modes with mode number n. However this cutoff seems unnatural since at $\alpha = 1$ it doesn't treat all of the massive modes on an equal footing. It leads to the same f_1 as the physical prescription.

In the appendices I also work out some finite-J corrections to giant magnons, since this is easily done with the algebraic curve. The main points to note are:

4. The exponent of the classical μ-term (36) depends on the "mass" of the giant magnon. This is just a result of embedding the well-known solutions into this spacetime, (44).

5. The exponents of the one-loop F-terms depend on the masses of the virtual particles. This can be thought of as being a consequence of the scaling of the AFS phase introduced by [15], which itself may be thought of as a result of the scaling of the time delay when magnon scattering happens on a sphere of smaller radius. Is has the effect that, unlike the $AdS_4 \times CP^3$ case, in general the bound-state and twice-wrapped contributions will not coincide [51]. Since each F-term depends only on one mass of virtual particle, these terms are unaffected by the choice of regularisation prescription [52,28].

6. Taking the $\alpha \to 1$ limit gives the same F-term corrections as those calculated directly from the algebraic curve for $AdS_3 \times S^3$, (42). This limit removes the trivial terms (corresponding to an S-matrix element of 1) arising from the fact that giant magnons on different spheres pass each other on the worldsheet without interacting.

Relation to other work

A recent paper by Sundin and Wulff [18] calculates, among other things, one-loop mass corrections to string states in the near-BMN limit of $AdS_3 \times S^3 \times S^3 \times S^1$. It is interesting to compare results since their paper works from the full Green–Schwartz action and thus includes all of the massless modes.

7. For the light bosons these mass corrections should agree with the small-p limit of the corrections to giant magnons [53], and indeed they match perfectly for both physical ("WS") and new ("AC") prescriptions, (31). These are given for $\alpha = \frac{1}{2}$ and $\alpha = 1$.

8. The correction for the heavy boson is computed by [18] at all α, and while using the physical sum gives a result consistent with the giant magnon's (26), using the new sum gives $c = \frac{1}{2\pi}\sin^2 2\phi \log(\sin^2 2\phi)$ rather than zero.[7] This seems strange as c should be universal.

   Note however that (as in [54]) the new prescription can only be implemented for the tadpole diagrams. For the bubbles the same loop momentum applies to two modes of different masses, in this case one of each mass of light mode. (Thus there is no issue at $\phi = \frac{\pi}{4}$, where this calculation gives $c = 0$.) The massless modes play no role in this calculation.

9. Virtual massless modes *do* appear to play a role for the corrections to the light bosons. However let me observe that simply deleting all diagrams containing them[8][9] does not change the result. This is an extremely naïve thing to do (since for instance the cubic interaction $\mathcal{L}_3$, [18]'s equation (3.3), has terms linear in the massless boson) but nevertheless perhaps interesting.

There are some points of overlap with other papers studying integrability in $AdS_3 \times S^3 \times S^3 \times S^1$ worth mentioning:

10. For the case $\alpha = 1$ (i.e. $AdS_3 \times S^3$) the energy corrections for giant magnons here agree with those of [11], who also used the algebraic curve formalism. In particular we agree that $c = 0$ in this limit.

11. As mentioned briefly in the introduction the recent papers [36] and [37] (which appeared while this was in preparation) each propose an S-matrix for this system. This should agree with terms in the F-term corrections calculated here, and a quick comparison shows agreement up to certain phases. See discussion in appendix A.

Finally, some comments on the closely related issues in $AdS_4 \times CP^3$. There, in all calculations to date one can use either the physical or the new prescription, and the change in the results is always equivalent to

$$c_{\text{phys}} = -\frac{\log 2}{2\pi}, \qquad c_{\text{new}} = 0$$

without running into the problems of points 1, 2 above. Nevertheless various arguments have been advanced for choosing one or the other. Below is a very brief list of these; most will also apply to the present $AdS_3 \times S^3 \times S^3 \times S^1$ case.

12. In favour of the physical sum, in [54] pointed out that it is difficult to see how to implement the new sum for Feynman diagrams containing modes of different masses in the same loop. (See also point 8.) More strongly, [56] uses a general condition from [57] that the vacuum energy should not depend on the topology when the soliton mass is zero. It would be interesting to see what this says about the present case.

13. In favour of the new sum, [29] argue that unitarity requires that the energy of two light modes near to their cutoff should correspond to that of a heavy mode near its cutoff.

---

[7] While this mode is not a cromulent giant magnon, since it is in AdS, one can nevertheless attempt a naïve treatment of a giant "4" mode by turning on $G_1 = \frac{-1}{2\sin^2\phi} G_{\text{mag}}$, $G_2 = -G_{\text{mag}}$ and $G_3 = \frac{-1}{2\cos^2\phi} G_{\text{mag}}$. Then the calculation of δE looks very much the same as that in section 4 above, and in particular is zero for the new sum.

[8] That is, delete all integrals $I_n^s(\ldots m_4)$ in $\mathcal{A}_B^2$ above (4.22) and $I_n^s(m_4)$ in $\mathcal{A}_T^i$ (4.23), where $m_4 \ll 1$ is [18]'s IR regulator for the massless modes. However for this to work we must use the canonical dimensionally regulated integrals with measure $\int d^d k/(2\pi)^d$ [55], rather than (4.10)'s $\int d^d k/(2\pi)^2$; this does not seem to affect final results such as (4.25).

[9] I am grateful to Per Sundin for discussions on this point.

(See also point 3.) Perhaps also under this heading it should be mentioned that [58] conjecture an all-loop $h(\lambda)$ consistent with their 4-loop weak-coupling result; this gives $c = 0$. And lastly, [59] might also be included (see also [60, 54, 18]) on the grounds that questions of the position of the heavy mode's pole vs. the two-particle cut are only subtle when $c = 0$.

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I would like to thank Diego Bombardelli, Justin David, Olof Ohlsson Sax, Bogdan Stefański, Per Sundin and Kostya Zarembo for discussions and correspondence, and IFT-UNESP São Paulo for hospitality while finishing this.

A. Finite-J Corrections

While this is somewhat aside from the paper, the algebraic curve can also be used to compute certain finite-size corrections. The classical μ-term is mostly just a check that things are working well. The one-loop F-terms will perhaps teach us something about the $\alpha \to 1$ limit.

Classical μ-term

The first class of finite-J corrections are μ-terms, suppressed by $e^{-m_r \Delta/E}$. The leading μ-term is the classical correction away from $J = \infty$, and from the sigma-model point of view we expect to get results almost identical to those for magnons in $\mathbb{R} \times S^2$ [61, 62] or dyonic magnons in $\mathbb{R} \times S^3$ [63–65] since the same string solutions can be embedded into this spacetime.[10] Nevertheless it is a check of the algebraic curve description.

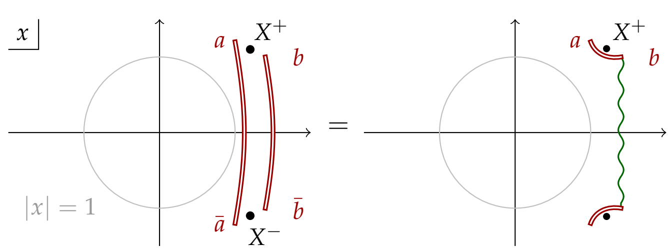
To work this out we must use a different algebraic curve solution, and following [64, 69, 67] we can use the following (approximate) form for the magnon resolvent:

$$G_{\text{finite}}(x) = -2i \log \left(\frac{\sqrt{x-a} + \sqrt{x-b}}{\sqrt{x-\bar{a}} + \sqrt{x-\bar{b}}} \right) \tag{34}$$

where the branch points are

$$a = \mathrm{X}^+ (1 + \tfrac{\delta}{2} e^{i\psi}), \qquad b = \mathrm{X}^+ (1 - \tfrac{\delta}{2} e^{i\psi})$$

and complex conjugates (with δ real). Recall that this arises from the two-cut solution, reorganising the two square root cuts (drawn) at the cost of producing a log cut (drawn) like this:



Thus $\delta \to 0$ gives us the $J = \infty$ giant magnon (23) above [48].

To use this resolvent here, we should set $G_1(x) = \frac{1}{2\sin^2\phi} [G_{\text{finite}}(x) - \frac{1}{2} G_{\text{finite}}(0)]$ in (9). Expanding the charges J', Q, and P in δ, the correction to the dispersion relation is obtained

---

[10] By contrast, in $AdS_4 \times CP^3$ only the non-dyonic solutions are the same [66], the dyonic solutions are new [67, 68] and known at finite J only in the algebraic curve language.

as follows:

$$\begin{aligned}\delta E^{\mu} &= \Delta - J' - \sqrt{Q^2 \sin^4\phi + 16 g^2 \sin^2 \frac{P}{2}} \\ &= \delta^2 \, \frac{g}{4} \, \cos(2\psi) \, \sin\frac{p}{2} + \mathcal{O}(\delta^4) \qquad \text{(non-dyonic).}\end{aligned}$$

The factor $\cos(2\psi)$ gives the effect of the angle between subsequent magnons, as we discussed in [67]. Next we must fix δ by demanding matching across the cut. To do this, use (7) in the form:

$$p_1(X^+ - i0) = p_1(X^+ + i0) - A_{1\ell}\, p_\ell(X^+ + i0) + 2\pi n \tag{35}$$

i.e.

$$G_{\text{finite}}(X^+ - i0) = (1 - 4\sin^2\phi)\, G_{\text{finite}}(X^+) - 4\sin^4\phi \,\frac{\Delta}{2g}\frac{X^+}{X^{+2} - 1} - 2\sin^2\phi \, 2\pi n.$$

Expanding and solving then gives

$$\begin{aligned}\delta^2 &= -16 (X^+ - X^-)^2 \exp -i\Big(\frac{\Delta \sin^2\phi}{g}\frac{X^+}{X^{+2}-1} + 2\pi n + 2\psi + \frac{\pi}{2\sin^2\phi}\Big) \\ &= 64 \, \exp\Big(\frac{-\Delta\sin^2\phi}{2g\,\sin^2\frac{p}{2}}\Big)\, \sin^2\frac{p}{2} \qquad \text{(non-dyonic).}\end{aligned}$$

The second line assumes that δ is real, which imposes $p + 2n\pi + 2\psi + \frac{1}{2}\pi \operatorname{cosec}^2\phi = 0$. Then setting $\psi = \frac{\pi}{2}$ the correction is

$$\delta E^{\mu} = -16 g \, e^{-\Delta\sin^2\phi \big/ 2g\sin^2\frac{p}{2}} \sin^3\frac{p}{2}. \tag{36}$$

This $\sin^3\frac{p}{2}$ behaviour matches what was given by AFZ [61]; the scaling of the exponent comes from placing this into a spheres of a different radius — see appendix C.

One-loop F-terms

The second class of finite-J corrections are F-terms, suppressed by both $e^{-m_r\Delta/g}$ and $1/g$ relative to the leading term E_0. These can readily be computed using the one-loop mode sum as above: they are simply the subsequent terms in the expansion of the cotangent in (25), for which the integral can be evaluated using the saddle point at $x = i$. Similar calculations were done by [49] in S^5 and [26,28,52] in CP^3.

The expansion is as follows:

$$\begin{aligned}\cot(\pi n)\, \pi n' &= \partial_x \log\big(\sin(\pi n)\big) \\ &= \pm i\pi n' + \partial_x \log\big(1 - e^{\mp i\, 2\pi n}\big) \\ &= \pm i\Big[\pi n' + e^{\mp i\, 2\pi n} 2\pi n' + \frac{e^{\mp 2i\, 2\pi n}}{2} 2\pi n' + \frac{e^{\mp 3i\, 2\pi n}}{3} 2\pi n' + \ldots\Big].\end{aligned}$$

In (25) above, the first term gave the infinite-volume one-loop correction. Subsequent terms can be written

$$\begin{aligned}\delta E^F &= \frac{-i}{2\pi}\int_{U_+} dx \sum_r (-1)^{F_r}\Big[e^{-i 2\pi n_r(x)} + \frac{e^{-2i\, 2\pi n_r(x)}}{2} + \frac{e^{-3i\, 2\pi n_r(x)}}{3} + \ldots\Big]\partial_x \Omega_r(x) \\ &= \sum_{\ell=1,2,3,\ldots}\frac{1}{\ell}\sum_{r=1,3,4.}\sqrt{\frac{m_r}{2\pi\kappa}}\, F^{(\ell)}_{m_r}(i)\end{aligned}$$

where we use the saddle point at $x = i$, and define $F_m^{(\ell)}(x) = \sum_{r:m_r=m} (-1)^{F_r} \exp(-i\,\ell\, 2\pi n_r(x)\,)$. This is the factor which in the Lüscher formula is $\sum_b (S_{b1}^{b1})^\ell$ [70]. (The Ω' factor is the Jacobian there.)

Here are some of the resulting integrands, taking as the classical solution the magnon in G_3 (i.e. the magnon in the sphere which survives at $\phi = 0$). First from the light modes $1, 1f, \bar{1}, \bar{1}f$ (in that order) writing just the $\ell = 1$ case:

$$F_{m_1}^{(1)} = \exp\Big(-\frac{\Delta \sin^2\phi}{g} \frac{i\,x}{x^2-1} \Big) \left[1 - \sqrt{\frac{X^-}{X^+}} \frac{x - X^+}{x - X^-} + 1 - \sqrt{\frac{X^+}{X^-}} \frac{1 - xX^-}{1 - xX^+} \right]. \tag{37}$$

Next from the corresponding modes of mass $\cos^2\phi$, the same as the giant magnon, namely $3, 3f, \bar{3}, \bar{3}f$

$$F_{m_3}^{(1)} = e^{-\frac{\Delta \cos^2\phi}{g} \frac{i\,x}{x^2-1}} \left[\frac{X^+}{X^-} \Big(\frac{x - X^-}{x - X^+} \Big)^2 - \sqrt{\frac{X^+}{X^-}} \frac{x - X^-}{x - X^+} + \frac{X^-}{X^+} \Big(\frac{1 - xX^+}{1 - xX^-} \Big)^2 - \sqrt{\frac{X^-}{X^+}} \frac{1 - xX^+}{1 - xX^-} \right]. \tag{38}$$

And finally from the heavy modes $4, 4f, \bar{4}, \bar{4}f$:

$$F_{m_4}^{(1)} = e^{-\frac{\Delta}{g} \frac{i\,x}{x^2-1}} \left[1 - \sqrt{\frac{X^+}{X^-}} \frac{x - X^-}{x - X^+} + 1 - \sqrt{\frac{X^-}{X^+}} \frac{1 - xX^+}{1 - xX^-} \right]. \tag{39}$$

Two comments on these:

- Note that as in $AdS_4 \times CP^3$ the modes of different masses lead to different factors in the exponential, but now there are three terms. These are separately finite and thus the regulator used is irrelevant. These exponential factors can perhaps be thought of as coming from the scaled σ_{AFS} as in (2.11) of [15].
- About the heavy modes, note also that their terms here are products of the constituent light modes', $4 = 1f + 3f$ etc. This is what we would expect if the S-matrices for these can be made by fusion, as in [51]. Unlike $AdS_4 \times CP^3$, the $\ell = 2$ term from a light mode wrapping twice will not coincide with the $\ell = 1$ heavy mode [51].

Comparison with S-matrices

The F-term formulae above should agree with certain diagonal elements of the recently published S-matrices for this system. This section aims to make some quick comparisons.[11]

Comparing first to Ahn and Bombardelli's [36], for the modes on the left the terms of their equation (2.14) we need are

$$\begin{aligned} S^{(33)}(p_1, p_2) &= S_0(p_1, p_2) \widehat{S}(p_1, p_2) \\ S^{(13)}(p_1, p_2) &= \widehat{S}(p_1, p_2) \end{aligned}$$

where $x_{p_1}^\pm \approx x$ for the virtual particle, $x_{p_2}^\pm = X^\pm$ for the physical giant magnon, and the matrix part is thus

$$\widehat{S} = \begin{cases} 1, & \text{bose-bose} \\ \dfrac{x - X^+}{x - X^-}, & \text{fermi-bose.} \end{cases}$$

[11] I am grateful to Diego Bombardelli and Olof Ohlsson Sax for discussing their results.

For the scalar factor S_0, only the classical AFS [71] term in the BES phase [33] matters, giving

$$S_0 = \sigma_{\text{AFS}}^2(p_1, p_2)\, e^{-ip_1+ip_2} = \left(\frac{x - 1/X^+}{x - 1/X^-}\right)^2 \frac{X^+}{X^-} e^{-2i\frac{x}{x^2-1}E/h} + \mathcal{O}\Big(\frac{1}{h}\Big).$$

Then since $X^+ = 1/X^- + \mathcal{O}(1/g)$ we have agreement with the first two terms of (37) and of (38), apart from phases $e^{-ip_2/2} = \sqrt{X^-/X^+}$ for the fermions ($1f$ and $3f$) which will arise from going to the string frame.

Note however that I have ignored here the phase $e^{-2i\frac{x}{x^2-1}E/h}$ which is part of σ_{AFS}. In the more familiar AdS_5 and AdS_4 cases the same power of σ_{AFS} appears in all terms, and changes the exponent $e^{-iq_\star L}$ from the Lüscher formula (with $L = J$) into $e^{-\Delta/h}$ from the algebraic curve. The absence of such a phase for $S^{(13)}$ is a feature of [36]'s S-matrix designed to match the Bethe equations of [15].

For the modes on the right, we need to look at

$$S^{(\bar{3}3)}(p_1, p_2) = \widetilde{S}_0(p_1, p_2)\widehat{S}(p_1, p_2)$$
$$S^{(\bar{1}3)}(p_1, p_2) = \widehat{S}(p_1, p_2).$$

Here $\widetilde{S}_0(p_1, p_2) = \sigma_{\text{AFS}}^{-2}(p_1, \bar{p}_2)\, e^{ip_1+ip_2}$ where the bar means $x^\pm_{\bar{p}} = 1/x^\pm_p$, and $\widehat{S}$ is as before. These match the last two terms of (37) and (38), up to the same two phase issues as for the left-hand modes.

Next consider the S-matrix given by Borsato, Ohlsson Sax and Sfondrini in [37]. This is written in terms of four unfixed phases, related by crossing relations. The coefficients relevant for (38) are A^{LL}_{pq}, B^{LL}_{pq}, A^{LR}_{pq}, C^{LR}_{pq}, and we should use the string frame expressions in appendix E. Then setting $x^\pm_q = x + \mathcal{O}(1/g)$ for the virtual particle, and $x^\pm_p = X^\pm$ for the physical giant magnon, the unfixed phases are

$$S^{LL}_{pq} = \sqrt{\frac{x_p^+}{x_p^-}}\,\frac{x - x_p^-}{x - x_p^+}\,\sigma_3(x, x_p^\pm), \qquad \tau^{LR}_{pq} = \sqrt{\frac{x_p^+}{x_p^-}}\,\frac{x - 1/x_p^+}{x - 1/x_p^-}\,\sigma_3(x, x_p^\pm).$$

Here some phase σ_3 is needed because the Lüscher formula gives $e^{-i\,m_r q_\star L}$ (with $q_\star = \frac{1}{h}\frac{x}{x^2-1}$) rather than the first factor in (38). If $L = J'$ then this could be provided (in this limit) by some power of the AFS phase.

In order to check crossing symmetry we need first (5.27):

$$S^{LR}_{pq} = \frac{1}{\zeta_{pq}}\tau^{LR}_{pq} = \sqrt{\frac{x_p^+}{x_p^-}}\left(\frac{x - 1/x_p^+}{x - 1/x_p^-}\right)^{3/2}\sigma_3.$$

Then using $x^\pm_{\bar{q}} = 1/x^\pm_q \approx 1/x$, we obtain

$$S^{LL}_{pq} S^{LR}_{p\bar{q}} = \sqrt{\frac{x_p^-}{x_p^+}}\sqrt{\frac{x - x_p^+}{x - x_p^-}}\sigma_3(x, x_p^\pm)\sigma_3(\tfrac{1}{x}, x_p^\pm).$$

Provided $\sigma_3(x)\sigma_3(\frac{1}{x}) = 1$ this is the inverse of the crossing relation (5.44).

For (37), where the virtual particle is of the opposite mass to the real particle, the relevant

terms are $\mathsf{A}_{pq}^{LL'}$, $\mathsf{B}_{pq}^{LL'}$, $\mathsf{A}_{pq}^{LR'}$, $\mathsf{C}_{pq}^{LR'}$, and the phases are

$$S_{pq}^{LL'} = \sqrt{\frac{x_p^-}{x_p^+}}\sigma_1, \qquad \tau_{pq}^{LR'} = \frac{x - 1/x_p^-}{x - 1/x_p^+}\sqrt{\frac{x_p^-}{x_p^+}}\sigma_1.$$

$$S_{pq}^{LR'} = \frac{1}{\zeta^{LR'}}\tau_{pq}^{LR'} = \sqrt{\frac{x - 1/x_p^-}{x - 1/x_p^+}}\sqrt{\frac{x_p^-}{x_p^+}}\sigma_1.$$

Then we obtain the inverse of the crossing relation (5.46), as long as $\sigma_1(x)\sigma_1(\frac{1}{x}) = 1$:

$$S_{pq}^{LL'} S_{p\bar{q}}^{LR'} = \sqrt{\frac{x - x_p^-}{x - x_p^+}}\sqrt{\frac{x_p^-}{x_p^+}}\sigma_1(x)\sigma_1(\tfrac{1}{x}).$$

B. Algebraic Curves for $AdS_3 \times S^3$ sans T^4

This appendix sets up the "T^4" case in exactly the same way as above, following [14]. The main reason for doing so is in order to calculate F-term corrections, to illustrate the limit $\alpha \to 1$.

The Cartan matrix is:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix} \otimes 1_{2\times 2}\,.$$

Here is the basis given by [14], with a minus inserted on the left-hand half this time, and the order of the index i chosen so that the quasimomenta $q_i(x)$ match those in [49, 42]:

$$\begin{array}{ccc c ccc|cc}
H_1 & H_2 & H_3 & & H_{\bar{1}} & H_{\bar{2}} & H_{\bar{3}} & & i \\
-1 & & & & & & & F & \hat{1} \\
-1 & 1 & & & & & & B & \tilde{1} \\
 & -1 & 1 & & & & & B & \tilde{4} \\
 & & 1 & & & & & F & \hat{4} \\
\hline
 & & & & 1 & & & F & \hat{2} \\
 & & & & 1 & -1 & & B & \tilde{2} \\
 & & & & & 1 & -1 & B & \tilde{3} \\
 & & & & & & -1 & F & \hat{3}
\end{array}$$

The inversion symmetry is the same as (5) above, or in terms of the q_i:

$$\hat{q}_1(\tfrac{1}{x}) = -\hat{q}_2(x), \qquad \hat{q}_3(\tfrac{1}{x}) = -\hat{q}_4(x)$$
$$\tilde{q}_1(\tfrac{1}{x}) = -\tilde{q}_2(x), \qquad \tilde{q}_3(\tfrac{1}{x}) = -\tilde{q}_4(x).$$

The vacuum is given by

$$\kappa = \frac{\Delta}{2g}\big(-1, 0, -1,\ 1, 0, 1\big)$$
$$q_i(x) = \frac{\Delta}{2g}\frac{x}{x^2-1}\big(1, 1, -1, -1,\ 1, 1, -1, -1\big).$$

We can again make modes by colouring in, this time 2 and $\bar{2}$ are the momentum-carrying

nodes:

$$
\begin{array}{lccc}
\text{Bosons:} & \oplus\!-\!\!\bullet\!\!-\!\oplus & (\tilde{1},\tilde{4}) & (\tilde{2},\tilde{3}) \\
 & \boldsymbol{\oplus}\!-\!\!\bullet\!\!-\!\boldsymbol{\oplus} & (\widehat{1},\widehat{4}) & (\widehat{2},\widehat{3}) \\
\text{Fermions:} & \boldsymbol{\oplus}\!-\!\!\bullet\!\!-\!\oplus & (\widehat{1},\tilde{4}) & (\widehat{2},\tilde{3}) \\
 & \oplus\!-\!\!\bullet\!\!-\!\boldsymbol{\oplus} & (\tilde{1},\widehat{4}) & (\tilde{2},\widehat{3})
\end{array}
\tag{40}
$$

where again $\bullet = 1$ on the left but $\bullet = -1$ on the right. Then the asymptotic behaviour of the modes is

$$
\begin{pmatrix}
\delta\widehat{q}_1 \\ \delta\widehat{q}_2 \\ \delta\widehat{q}_3 \\ \delta\widehat{q}_4 \\ \hline \delta\tilde{q}_1 \\ \delta\tilde{q}_2 \\ \delta\tilde{q}_3 \\ \delta\tilde{q}_4
\end{pmatrix}
\to \frac{1}{2gx}
\begin{pmatrix}
\delta\Delta + N_{\widehat{1}\widehat{4}} + N_{\widehat{1}\tilde{4}} \\
\delta\Delta + N_{\widehat{2}\widehat{3}} + N_{\widehat{2}\tilde{3}} \\
-\delta\Delta - N_{\widehat{2}\widehat{3}} - N_{\tilde{2}\widehat{3}} \\
-\delta\Delta - N_{\widehat{1}\widehat{4}} - N_{\tilde{1}\widehat{4}} \\ \hline
-N_{\tilde{1}\tilde{4}} \quad - N_{\tilde{1}\widehat{4}} \\
-N_{\tilde{2}\tilde{3}} \quad - N_{\tilde{2}\widehat{3}} \\
+N_{\tilde{2}\tilde{3}} \quad + N_{\widehat{2}\tilde{3}} \\
+N_{\tilde{1}\tilde{4}} \quad - N_{\widehat{1}\tilde{4}}
\end{pmatrix}
+ \dots
\tag{41}
$$

which matches the 1st & 4th columns of [42]'s (A.9), apart from normalisation of Δ. As expected this is very much like $AdS_5 \times S^5$ with the modes connecting left and right turned off. Comparing this with (19) at $\phi = 0$, the bosons match up perfectly but for the fermions things aren't so simple.

F-term Corrections

Using a giant magnon $G_2(x) = G_{\text{mag}}(x)$, i.e. a giant $(\tilde{1},\tilde{4})$ mode, here are some of the integrands $F^{(\ell)}(x) = \sum_{ij}(-1)^{F_{ij}} \exp(-i\ell(q_i - q_j))$, showing terms in the same order as (40) above:

$$
\begin{aligned}
F^{(1)}_{\text{left}} &= e^{-\frac{\Delta}{g}\frac{ix}{x^2-1}} \left[\frac{X^+}{X^-}\left(\frac{x-X^-}{x-X^+}\right)^2 + 1 - \frac{x-X^-}{x-X^+}\sqrt{\frac{X^+}{X^-}} - \frac{x-X^-}{x-X^+}\sqrt{\frac{X^+}{X^-}} \right] \\
F^{(1)}_{\text{right}} &= e^{-\frac{\Delta}{g}\frac{ix}{x^2-1}} \left[\frac{X^-}{X^+}\left(\frac{1-xX^+}{1-xX^-}\right)^2 + 1 - \frac{1-xX^+}{1-xX^-}\sqrt{\frac{X^-}{X^+}} - \frac{1-xX^+}{1-xX^-}\sqrt{\frac{X^-}{X^+}} \right]
\end{aligned}
\tag{42}
$$

These line up with the $AdS_3 \times S^3 \times S^3$ ones above, in total

$$
F^{(1)}_{m_3}\Big|_{\phi=\pi/2} + F^{(1)}_{m_4} = F^{(1)}_{\text{left}} + F^{(1)}_{\text{right}}
$$

but also term by term.

C. Worldsheet Theory

The bosonic action in conformal gauge (and setting $\alpha' = 1$) is

$$
\mathcal{S} = \int \frac{d\tau d\sigma}{4\pi} \left(R^2 \partial_\mu \overline{Z} \cdot \partial^\mu Z + \frac{R^2}{\cos^2\phi}\partial_\mu \overline{X} \cdot \partial^\mu X + \frac{R^2}{\sin^2\phi}\partial_\mu \overline{Y} \cdot \partial^\mu Y + R^2 \partial_\mu \psi \partial^\mu \psi \right)
$$

where $|Z|^2 = -1$ describes AdS_3 embedded in $\mathbb{C}^{1,1}$, and $|X|^2 = |Y|^2 = 1$ describe the two spheres each in $\mathbb{C}^2$. The equations of motion are

$$
\begin{aligned}
0 &= \partial_\mu \partial^\mu Z + \left(\partial_\mu \overline{Z} \cdot \partial^\mu Z\right) Z & \qquad 0 &= \partial_\mu \partial^\mu Y + \left(\partial_\mu \overline{Y} \cdot \partial^\mu Y\right) Y \\
0 &= \partial_\mu \partial^\mu X + \left(\partial_\mu \overline{X} \cdot \partial^\mu X\right) X & \qquad 0 &= \partial_\mu \partial^\mu \psi
\end{aligned}
$$

with the four components coupled only through the Virasoro constraints

$$
\begin{aligned}
0 = \partial_\tau \overline{Z} \cdot \partial_\tau Z + \partial_\sigma \overline{Z} \cdot \partial_\sigma Z &+ \frac{1}{\cos^2\phi}\left(\partial_\tau \overline{X} \cdot \partial_\tau X + \partial_\sigma \overline{X} \cdot \partial_\sigma X\right) \\
&+ \frac{1}{\sin^2\phi}\left(\partial_\tau \overline{Y} \cdot \partial_\tau Y + \partial_\sigma \overline{Y} \cdot \partial_\sigma Y\right) + \left(\partial_\tau \psi \cdot \partial_\tau \psi + \partial_\sigma \psi \cdot \partial_\sigma \psi\right) \\
0 = \partial_\tau \overline{Z} \cdot \partial_\sigma Z &+ \frac{1}{\cos^2\phi}\left(\partial_\tau \overline{X} \cdot \partial_\sigma X\right) + \frac{1}{\sin^2\phi}\left(\partial_\tau \overline{Y} \cdot \partial_\sigma Y\right) + \text{c.c.} \quad + \partial_\tau \psi \cdot \partial_\sigma \psi.
\end{aligned}
$$

The global charges are

$$
\begin{aligned}
\Delta &= R^2 \int_{-L}^{L} \frac{d\sigma}{2\pi} \operatorname{Im}(\overline{Z}_0 \cdot \partial_\tau Z_0) \\
J_X &= \frac{R^2}{\cos^2\phi} \int_{-L}^{L} \frac{d\sigma}{2\pi} \operatorname{Im}(\overline{X}_1 \cdot \partial_\tau X_1), \qquad J_Y = \frac{R^2}{\sin^2\phi} \int_{-L}^{L} \frac{d\sigma}{2\pi} \operatorname{Im}(\overline{Y}_1 \cdot \partial_\tau Y_1) \\
J' &= \cos^2\phi \, J_X + \sin^2\phi \, J_Y
\end{aligned}
$$

and

$$
P = \int d\sigma \operatorname{Im}(\partial_\sigma \log X_1) + \int d\sigma \operatorname{Im}(\partial_\sigma \log Y_1).
$$

The spinning string solution studied by [16] is

$$
Z_0 = e^{i\kappa\tau} \cosh\rho(\sigma), \qquad Z_1 = e^{i\omega\tau} \sinh\rho(\sigma), \qquad X_1 = e^{i\nu_+\tau}, \qquad Y_1 = e^{i\nu_-\tau} \tag{43}
$$

for which the Virasoro constraint gives

$$
0 = -\kappa^2 \cosh^2\rho + \omega^2 \sinh^2\rho + \rho'^2 + \frac{1}{\cos^2\phi}\nu_+^2 + \frac{1}{\sin^2\phi}\nu_-^2 \,.
$$

Here we consider only the case $\nu_+ = \cos^2\phi \, \nu$, $\nu_- = \sin^2\phi \, \nu$, giving $J_X = J_Y = \nu L R^2 / 2\pi$. This reduces to the supersymmetric BMN point particle when $\rho = 0$, and $\kappa = \nu$ [43], see also [72] for this background. At $\phi = 0$ this is stationary on the Y sphere, and the last term drops out of the Virasoro constraint.

Now consider placing magnons into this space. Treating immediately the finite-J case, let $X_{\text{fin}}(\sigma, \tau)$ be a solution in $\mathbb{R} \times S^3$ in conformal gauge and with $t = \tau$: it satisfies $\partial_\tau \overline{X} \cdot \partial_\tau X + \partial_\sigma \overline{X} \cdot \partial_\sigma X = 1$. Let $2L_{\text{fin}}$ be the periodicity in σ (i.e. the distance between two cusps). Writing charges using this as Δ_{fin} (and $J_{\text{fin}} = J'|_{\phi=0}$) it has dispersion relation

$$
\Delta_{\text{fin}} - J_{\text{fin}} = 4g \sin\frac{p}{2}\left(1 - 4\,\sin^2\frac{p}{2}\, e^{-2\Delta_{\text{fin}} / 4g \sin\frac{p}{2}} + \dots\right).
$$

The solution at general ϕ is

$$
\begin{aligned}
X(\sigma, \tau) &= X_{\text{fin}}(\cos^2\phi \, \sigma, \cos^2\phi \, \tau) \\
Y_1(\sigma, \tau) &= e^{i \sin^2\phi \, \tau}, \qquad Y_2 = 0
\end{aligned} \tag{44}
$$

with $\cos^2\phi\, L = L_{\rm fin}$, thus $\Delta = \frac{1}{\cos^2\phi}\Delta_{\rm fin}$. This has $P = p$ and

$$\Delta - J' = \Delta_{\rm fin} - J_{\rm fin} = 4g \sin\frac{p}{2}\left(1 - 4\sin^2\frac{p}{2}\, e^{-2\Delta\cos^2\phi \big/ 4g\sin\frac{p}{2}} + \dots\right). \tag{45}$$

The exponent in the finite-J correction is exactly what we saw in the algebraic curve calculation (36), apart from here considering a magnon in the other sphere i.e. a giant "3" mode.

Other solutions can be similarly embedded, in particular:

- We can use the same scattering solutions as usual, [73], within one sphere. The time delay for scattering solutions is defined like this (initially on a unit sphere):

  $$X_{\rm scat}(\sigma,\tau) = \begin{cases} X_{\rm mag}(\sigma,\tau), & \sigma,\tau \to -\infty \\ X_{\rm mag}(\sigma,\tau - \Delta\tau_{\rm mag}), & \sigma,\tau \to +\infty. \end{cases}$$

  It is clear that embedding this solution into the S^3_+ sphere via (44) will give us a time delay $\Delta\tau = \frac{1}{\cos^2\phi}\tan\frac{p}{2}\log(\cos^2\frac{p}{2})$ scaled from the usual (centre of mass frame) delay [30].

- We can embed a different giant magnon into each sphere, and they don't talk to each other at all. Thus we would expect the relevant terms in the S-matrix to be 1, and this is exactly what we saw in (37) above: the trivial entries there correspond to a physical 3 mode and a virtual 1 or $\bar{1}$.

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